

Star-product in the presence of a monopole

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Abstract

We present a deformed \star -product for a particle in the presence of a magnetic monopole. The product is obtained within a self-dual quantization-dequantization scheme, with the correspondence between classical observables and operators defined with the help of a quaternionic Hilbert space, following work by Emch and Jadczyk. The resulting product is well defined for a large class of complex functions and reproduces (at first order in \hbar) the Poisson structure of the particle in the monopole field. The product is associative only for quantized monopole charges, thus incorporating Dirac's quantization requirement.

1 Introduction

In this paper we perform the deformation quantization for the algebra of functions on the phase space of an electrically charged particle in the field of a magnetic monopole at the origin; to be precise, the phase space is determined by the asymptotics of the 't Hooft–Polyakov monopole [2]. The deformation is therefore that of complex-valued functions on $(\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$ in the presence of the following Poisson structure:

$$\begin{aligned}\{p_i, x^j\} &= \delta_i^j \\ \{x^i, x^j\} &= 0 \\ \{p_i, p_j\} &= \frac{g}{2} \varepsilon_{ijk} \frac{x^k}{|x|^3},\end{aligned}\tag{1.1}$$

where the quantity g is the product of the monopole magnetic charge and the electric charge of a test particle. Inversion of the Poisson bracket gives a symplectic two-form

$$\omega = dx^i \wedge dp_i + \frac{g}{2} \varepsilon_{ijk} \frac{x^i}{|x|^3} dx^j \wedge dx^k.\tag{1.2}$$

Existence of a formal \star -product is of course ensured by general theorems for Poisson manifolds [8], but the aim of the paper is to follow as close as possible the Wigner–Weyl–Grönewold–Moyal quantization/dequantization program [13], in order to exhibit the product explicitly. For a modern treatment of this program, see [5, 6]. In such a scheme unitary operators are associated to translations in phase space, while commutation relations reproduce the exponentiated version of the Poisson structure. In this way not only the commutation relations of the operators associated to the x ’s and p ’s reproduce (up to $i\hbar$) the corresponding Poisson brackets, but one is at the same time assured of that the product is well defined on a large class of functions.

We will therefore seek a generalization of the Weyl map, which associates operators to functions. To the purpose we use a remarkable result by Emch and Jadczyk [4], based on quaternionic quantum mechanics. Here, however, quaternions and quaternion Hilbert modules are simply a device to build a \star -product, which in the end acts between *complex* functions. It is possible to adapt the construction for a deformed product between quaternion-valued functions, but we do not discuss this issue in the present paper.

The paper is organized as follows. In Section 2 we review Weyl systems and the Weyl–Wigner quantization-dequantization maps for ordinary quantum mechanics, which brings to the definition of the Moyal \star -product. In Section 3 we give an account of the Emch–Jadczyk construction, providing a setting for the quantization of a particle in the field of a magnetic monopole in terms of quaternionic quantum mechanics. In Section 4 we propose a generalized Weyl system in this context. Finally, in Section 5 we exhibit a generalized Weyl–Wigner construction, with a quantization and a dequantization map. This yields a description of the quantum particle-monopole system in terms of complex functions, with a new noncommuting \star -product. We add some concluding remarks.

2 The Weyl–Wigner–Grönewold–Moyal formalism

Schematically, the quantization of a particle with the usual Poisson canonical structure on the phase space $T^*\mathbb{R}^3$ (relations (1.1) with $g = 0$) goes as follows. Let us denote by $\tilde{f}(\eta, \xi)$ the Fourier transform of the phase-space function $f(x, p)$. To f it is associated an operator by performing the inverse transform and inserting in the integral a unitary operator family $\hat{T}(\eta, \xi)$ instead of $e^{-i(\eta \cdot p + \xi \cdot x)}$:

$$\hat{f} = \hat{\mathcal{W}}(f) = \frac{1}{(2\pi)^3} \int d\eta \, d\xi \, \hat{T}(\eta, \xi) \tilde{f}(\eta, \xi). \quad (2.1)$$

The family \hat{T} is in turn given by $\hat{T}(\eta, \xi) := e^{i(\eta \cdot \hat{P} + \xi \cdot \hat{X})}$, where \hat{X}^i and \hat{P}_j are the usual position and momentum operators. This Weyl map $f \rightarrow \hat{f}$ is well defined and invertible for large classes of functions; for example it associates Hilbert–Schmidt operators to square integrable functions [1, 12]. The unitary operators \hat{T} obey

$$\hat{T}(\eta, \xi) \hat{T}(\eta', \xi') = \hat{T}(\eta + \eta', \xi + \xi') e^{\frac{i}{2}(\eta \cdot \xi' - \eta' \cdot \xi)}. \quad (2.2)$$

or, which is the same,

$$\hat{T}(\eta, \xi) \hat{T}(\eta', \xi') \hat{T}^\dagger(\eta, \xi) \hat{T}^\dagger(\eta', \xi') = e^{i(\eta \cdot \xi' - \eta' \cdot \xi)} \hat{I}, \quad (2.3)$$

and we recognize the canonical symplectic form in the exponential. They yield a ray representation of the Euclidean group in 3 dimensions, what is called a Weyl system built on the symplectic vector space $(T^*\mathbb{R}^3, \omega)$, with the usual canonical commutation relations between coordinates and momenta descending from (2.3).

The inverse of the Weyl map, defined on suitable domains, is usually called the Wigner map [14]. It is convenient to express both the Weyl map and its inverse in terms of two operators, $\hat{\Omega}$, $\hat{\Gamma}$, respectively called *quantizer* and *dequantizer*:

$$\begin{aligned} \hat{f} &= \hat{\mathcal{W}}(f) = \frac{1}{(2\pi)^3} \int dx \, dp \, \hat{\Omega}(x, p) f(x, p) \\ f(x, p) &= \mathcal{W}^{-1}(\hat{f}) = \text{Tr} \, \hat{\Gamma}(x, p) \hat{f}. \end{aligned} \quad (2.4)$$

For the canonical case, up to constant factors depending on the normalization conventions,

$$\hat{\Omega}(x, p) = \hat{\Gamma}(x, p) = \int d\eta \, d\xi \, \hat{T}(\eta, \xi) e^{-i(\eta \cdot p + \xi \cdot x)}. \quad (2.5)$$

Notice however that there exist well defined quantization-dequantization maps for which the two operators are different [9, 10]. Functions associated to operators through the dequantization map, often called symbols, are actually noncommuting: they reproduce the noncommutativity of the operators by means of a noncommutative (star) product known as the Grönewold–Moyal product [7, 11]:

$$f \star g = \mathcal{W}^{-1} \left(\hat{\mathcal{W}}(f) \hat{\mathcal{W}}(g) \right). \quad (2.6)$$

This may be written in terms of an integral kernel, in turn completely specified by the operators $\hat{\Omega}$ and $\hat{\Gamma}$

$$f \star g(x, p) = \text{Tr } \hat{f} \hat{g} \hat{\Gamma}(x, p) = \int dx' dp' dx'' dp'' f(x', p') g(x'', p'') K(x', p'; x'', p''; x, p) \text{ with} \\ K(x', p'; x'', p''; x, p) = \text{Tr } \hat{\Omega}(x', p') \hat{\Omega}(x'', p'') \hat{\Gamma}(x, p) = C e^{2i[x(p'-p'')+x'(p''-p)+x''(p-p')]} \quad (2.7)$$

with C a normalization constant. This associative and noncommutative product is well defined also for polynomials and reproduces the canonical commutation relations through the so called Moyal bracket:

$$x^i \star p_j - p_j \star x^i = i\delta_j^i, \quad x^i \star x^j - x^j \star x^i = p_i \star p_j - p_j \star p_i = 0. \quad (2.8)$$

We want to generalize this construction to the particle/monopole system. We will need therefore the analogue of the operator \hat{T} (a generalized Weyl system), and a quantization map and a dequantization map. A generalized Weyl system was proposed by us in [3].

3 The construction by Emch and Jadczyk

Quaternionic quantum mechanics is buttressed by the observation that to describe particles with an inner structure it is necessary to consider sections of some hermitian complex vector bundle. In this philosophy, Emch and Jadczyk propose in [4] that a quantum particle in the field of the monopole be described by square integrable sections of a hermitian quaternionic line bundle over $\mathbb{R}^3 - \{0\}$. In that setting all operators, the generators of the generalized Weyl system in particular, are quaternionic valued. As announced earlier, we use their construction as an intermediate step to build a \star -product where operator symbols are genuinely complex-valued functions. We adopt the following notations. Quaternionic units are $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $\mathbf{e}_0 = 1$ with

$$\mathbf{e}_i \mathbf{e}_j = -\delta_{ij} \mathbf{e}_0 + \varepsilon_{ijk} \mathbf{e}_k, \quad i = 1, \dots, 3 \quad (3.1)$$

and the involution $\mathbf{e}_0^* = \mathbf{e}_0$, $\mathbf{e}_i^* = -\mathbf{e}_i$. All complex-valued quantities are in normal typeface: α, f, \dots ; all quaternionic quantities are in **sans serif**: $\mathbf{q}, \mathbf{f}, \mathbf{j}$; operators are distinguished by the usual hat $\hat{}$. Borrowing the representation of imaginary quaternions by means of 2×2 skew-hermitian matrices, we may write (when convenient)

$$\mathbf{e}_0 = \sigma_0, \quad \mathbf{e}_i = -i\sigma_i, \quad \text{and} \quad \mathbf{f}(x) = f^0(x)\mathbf{e}_0 + f^i(x)\mathbf{e}_i. \quad (3.2)$$

Let \mathcal{H} be the quaternionic Hilbert module formed by sections $\Psi(x)$ with the usual quaternionic-valued inner product

$$\langle \Psi, \Phi \rangle = \int dx \Psi(x)^* \Phi(x) \quad (3.3)$$

and the associated real norm. The elements of the Hilbert module behave as a vector space under multiplication by quaternions (numbers) from the *right*, while linear operators

act on the *left*. Among the operators there are of course also the quaternionic valued functions which act multiplicatively: $\hat{\mathbf{f}}\Psi(x) = \mathbf{f}(x)\Psi(x)$. The quaternionic line bundle may be considered as an associated bundle with structure group $SU(2)$. In order to lift vector fields from \mathbb{R}^3 to the total space of the vector bundle one has to introduce a connection, that is, a procedure to lift vector fields to horizontal vector fields, which takes into account the presence of the monopole. This is the *gauge potential*

$$A = g \frac{[\mathbf{e} \cdot x, \mathbf{e} \cdot dx]}{|x|^2}, \quad (3.4)$$

with the square bracket indicating the antisymmetrized product and g denoting the product of the electric and magnetic charges. We allow for the possibility that g be different from 1, which slightly generalizes the Emch and Jadczyk construction, and will clearly show how the monopole charge quantization emerges. The origin of the choice (3.4) may be traced back to the Hopf fibration. Since $\mathbb{S}^2 \times \mathbb{R}_+ = \mathbb{R}^3 - \{0\}$, we may define a lifting which would consider wave functions as fields transforming covariantly under the rotation group, whose action in the inner space is by means of $SU(2)$. Given any $u \in \mathbb{S}^2$, the translation $u \cdot \partial/\partial x$ lifts to the quaternionic-valued differential operator

$$\nabla_u = \mathbf{e}_0 u \cdot \frac{\partial}{\partial x} + \frac{g}{2} \frac{[\mathbf{e} \cdot x, \mathbf{e} \cdot u]}{|x|^2} = \mathbf{e}_0 u \cdot \frac{\partial}{\partial x} + \frac{g}{2} \mathbf{e} \cdot \frac{u \wedge x}{|x|^2}. \quad (3.5)$$

We use the notation ∇_i for the covariant derivatives along the cartesian basis vectors. It may be verified that it obeys $[\nabla_i, x_j] = \delta_{ij} \mathbf{e}_0$; thus it generates translations on configuration space. Moreover it transforms as a vector under rotations: $[M_i, \nabla_j] = -\varepsilon_{ijk} \nabla_k$, with $M_i = \varepsilon_{ijk} x_j \partial_k - \mathbf{e}_i/2$. Central to the construction is the following imaginary unit introduced by Emch and Jadczyk:

$$\mathbf{j}(x) = \frac{\mathbf{e} \cdot x}{|x|}; \quad \mathbf{j}^2 = -\mathbf{e}_0, \quad \mathbf{j}^* = -\mathbf{j}. \quad (3.6)$$

It is also rotationally invariant (we regard the \mathbf{e} 's as transforming like the components of a vector under rotations). Associated with \mathbf{j} , consider the linear operator $\hat{\mathbf{J}}$, commuting with translations:

$$\hat{\mathbf{J}}\Psi(x) := \mathbf{j}(x)\Psi(x). \quad (3.7)$$

Infinitesimal translations do not commute among themselves, their commutator being the monopole field

$$[\nabla_i, \nabla_j] = -\frac{1}{2} g \varepsilon_{ijk} \frac{x^k}{|x|^3} \hat{\mathbf{J}}. \quad (3.8)$$

Finite translations $\hat{\mathbf{U}}(\eta)$ generated by the operator ∇_η , for $\eta \in \mathbb{R}^3 - \{0\}$, acquire a quaternionic phase factor. We have indeed

$$\hat{\mathbf{U}}(\eta)\Psi(x) = \mathbf{w}(\eta, x - \eta)\Psi(x - \eta), \quad (3.9)$$

where $\mathbf{w}(\eta, x)$, for every η not collinear with x , is given by the quaternion

$$\mathbf{w}(\eta, x) = \exp\left(\mathbf{j}(x \wedge \eta) \frac{g\alpha}{2}\right), \quad (3.10)$$

with α the angle between x and $x + \eta$. This may be checked directly, on the strength of

$$\nabla_\eta \Psi(x) = \lim_{t \rightarrow 0} \left(\frac{d}{dt} (\hat{U}(t\eta) \Psi)(x) \right) \text{ and } \lim_{t \rightarrow 0} \frac{d}{dt} \mathbf{w}(t\eta, x - t\eta) = \frac{g}{2} \mathbf{e} \cdot \frac{x \wedge \eta}{|x|^2}. \quad (3.11)$$

Some easily verifiable properties of \mathbf{w} , satisfied almost everywhere, are:

1. $\mathbf{w}(0, x) = 1$
2. $\mathbf{w}(\eta, x) \mathbf{w}^*(\eta, x) = 1$
3. $\mathbf{w}^*(\eta, x) = \mathbf{w}(-\eta, x + \eta)$
4. $\mathbf{w}(t\eta, x + s\eta) \mathbf{w}(s\eta, x) = \mathbf{w}((s + t)\eta, x)$.

with t, s real parameters. We also consider the multiplication operators $\hat{W}(\eta) \Psi(x) := \mathbf{w}(\eta, x) \Psi(x)$. Operators $\hat{U}(\eta)$ may be written as the product of ordinary translations in \mathbb{R}^3 and the $\hat{W}(\eta)$:

$$\hat{U}(\eta) = \hat{V}(\eta) \hat{W}(\eta) \quad \text{with} \quad \hat{V}(\eta) \Psi(x) = \Psi(x - \eta). \quad (3.12)$$

The operator ∇ is skew-hermitian. Let us define the hermitian operators

$$\hat{P}_i = -\hat{J} \nabla_i = -\nabla_i \hat{J}. \quad (3.13)$$

We have

$$[\hat{P}_i, \hat{P}_j] = \frac{1}{2} \varepsilon_{ijk} g \frac{x^k}{|x|} \hat{J}. \quad (3.14)$$

The \hat{P}_i are generators of translations in the quaternionic Hilbert space. Notice that the two summands in \hat{P}_i do not commute. Finally,

$$\hat{U}(\eta) \hat{U}(\eta') \Psi(x) = \mathbf{w}(\eta; x - \eta) \mathbf{w}(\eta'; x - \eta - \eta') \Psi(x - \eta - \eta'). \quad (3.15)$$

The key result of Emch and Jadczyk is that this may be written in terms of $\hat{U}(\eta + \eta')$:

$$\hat{U}(\eta) \hat{U}(\eta') = \hat{U}(\eta + \eta') \hat{M}(\eta, \eta'). \quad (3.16)$$

by means of multiplication operator,

$$\hat{M}(\eta, \eta') \Psi(x) := \mathbf{m}(\eta, \eta'; x) \Psi(x) \quad (3.17)$$

with $\mathbf{m}(\eta, \eta'; x) = \mathbf{w}^*(\eta + \eta', x) \mathbf{w}(\eta, x + \eta') \mathbf{w}(\eta', x)$. Since $\mathbf{w}(0, x) = 1$ and $\mathbf{w}(\eta, x - \eta) = \mathbf{w}^*(-\eta, x)$ we have that

$$\mathbf{m}(\eta; -\eta; x) = 1. \quad (3.18)$$

The quantity \mathbf{m} can be expressed in exponential form [4]:

$$\mathbf{m}(\eta, \eta'; x) = \exp \left(\frac{g}{4} \mathbf{j}(x) \varepsilon_{ijk} \frac{x^i}{|x|^3} \eta^j \eta'^k \right) \quad (3.19)$$

where we recognize the flux of the monopole field through the flat triangle with vertices $(x, x + \eta, x + \eta + \eta')$. This result may be easily obtained by direct calculation, observing that (3.16) implies

$$\hat{M}(\eta, \eta') = \hat{U}^{-1}(\eta + \eta') \hat{U}(\eta) \hat{U}(\eta') \quad (3.20)$$

with $\hat{U}(\eta)$ in the form $\hat{U}(\eta) = \exp(\hat{J}\eta \cdot \hat{P})$ and repeatedly using the commutation relation (3.14). The associativity condition $(\hat{U}(\eta)\hat{U}(\eta'))\hat{U}(\eta'') = \hat{U}(\eta)(\hat{U}(\eta')\hat{U}(\eta''))$ requires

$$\hat{M}(\eta, \eta')\hat{M}(\eta + \eta', \eta'') = \hat{M}(\eta, \eta' + \eta'')\hat{M}(\eta', \eta'').$$

This last equality is satisfied only if the flux through the tetrahedron identified by the vectors $x, x + \eta, x + \eta + \eta', x + \eta + \eta' + \eta''$ is a multiple integer of 2π , that is only if g is an integer. This is the celebrated *quantization condition*. We see that the construction will yield an associative algebra only for systems which respect it. Hereafter for simplicity of notation we will consider the case $g = 1$.

Summarizing, the Emch–Jadczyk here reviewed provides a unitary representation for noncommuting translations in terms of quaternionic valued operators acting on quaternionic wave functions. This is the first stone of our construction.

4 The monopole Weyl system

On the quaternionic Hilbert module \mathcal{H} , we consider the six operators

$$\begin{aligned} \hat{X}^i \Psi(x) &= x^i \Psi(x), \\ \hat{P}_j \Psi(x) &= -(\hat{J} \nabla_j \Psi)(x), \end{aligned} \quad (4.1)$$

with commutation relations

$$\begin{aligned} [\hat{P}_i, \hat{X}_j] &= -\hat{J} \delta_{ij}, \\ [\hat{P}_i, \hat{P}_j] &= \frac{1}{2} g \hat{J} \varepsilon_{ijk} \frac{x^k}{|x|^3}, \\ [\hat{X}_i, \hat{X}_j] &= 0. \end{aligned}$$

They may be regarded as a deformation of the Euclidean algebra in 3 dimensions, although do not define a Lie algebra anymore. It is however possible to exhibit a unitary representation of the relations, what we call a generalized Weyl system [3]. This is provided by the operator family

$$\hat{T}(\alpha) = e^{\hat{J}[\eta \cdot \hat{P} + \xi \cdot \hat{X}]} = e^{\hat{J}\eta \cdot \hat{P}} e^{\hat{J}\xi \cdot \hat{X}} e^{\frac{1}{2} \eta^i \xi^j [\hat{P}^i, \hat{X}^j]} = e^{\hat{J}\eta \cdot \hat{P}} e^{\hat{J}\xi \cdot \hat{X}} e^{-\frac{1}{2} \hat{J}\eta \cdot \xi} = e^{\hat{J}\xi \cdot \hat{X}} e^{\hat{J}\eta \cdot \hat{P}} e^{\frac{1}{2} \hat{J}\eta \cdot \xi}, \quad (4.2)$$

with $\alpha = (\eta, \xi)$. Remember that $\exp(\hat{J}\eta \cdot \hat{P}) \equiv \hat{U}(\eta)$. We have then

$$\hat{T}(\alpha) \Psi(x) = e^{\hat{J}\eta \cdot \hat{P}} e^{\hat{J}\xi \cdot \hat{X}} e^{-\frac{1}{2} \hat{J}\eta \cdot \xi} \Psi(x) = w(\eta; x - \eta) e^{j(x-\eta)\xi \cdot (x-\eta)} e^{-\frac{1}{2} j(x-\eta)\eta \cdot \xi} \Psi(x - \eta), \quad (4.3)$$

but also

$$\hat{T}(\alpha)\Psi(x) = e^{\hat{J}\xi\cdot\hat{X}}e^{\hat{J}\eta\cdot\hat{P}}e^{\frac{1}{2}\hat{J}\eta\cdot\xi}\Psi(x) = e^{j(x)\xi\cdot x}\mathbf{w}(\eta; x - \eta)e^{\frac{1}{2}j(x-\eta)\eta\cdot\xi}\Psi(x - \eta). \quad (4.4)$$

We compute

$$\hat{T}(\alpha)\hat{T}(\beta) = e^{\hat{J}[\eta\cdot\hat{P}+\xi\cdot\hat{X}]}e^{\hat{J}[\eta'\cdot\hat{P}+\xi'\cdot\hat{X}]} = e^{\hat{J}\eta\cdot\hat{P}}e^{\hat{J}\xi\cdot\hat{X}}e^{\hat{J}\eta'\cdot\hat{P}}e^{\hat{J}\xi'\cdot\hat{X}}e^{-\frac{1}{2}\hat{J}(\eta\cdot\xi+\eta'\cdot\xi')}. \quad (4.5)$$

On using (3.16) and (4.2), we arrive at the sought for generalization of (2.3):

$$\hat{T}(\alpha)\hat{T}(\beta) = \hat{T}(\alpha + \beta)\hat{M}(\eta, \eta') \exp\left(\frac{1}{2}\hat{J}(\eta\cdot\xi' - \eta'\cdot\xi)\right). \quad (4.6)$$

Like its usual counterpart (2.2), this generalized Weyl system provides a projective representation of the translation group, but in this case there are two phases. One is present also in the usual quantization scheme and gives the noncommutativity of positions and momenta —here however with the imaginary unit replaced by the quaternionic radial function $j(x)$. The factor \hat{M} instead contains the information on the noncommutativity of the translations.

On using the identity

$$1 = \hat{T}(-\beta)\hat{T}(-\alpha)\hat{T}(\alpha)\hat{T}(\beta) = \hat{T}(-\alpha - \beta)\hat{M}(-\eta', -\eta)\hat{T}(\alpha + \beta)\hat{M}(\eta, \eta'), \quad (4.7)$$

and observing that, from (4.6)

$$\begin{aligned} & \hat{T}(-\alpha)\hat{T}(-\beta)\hat{T}(\alpha)\hat{T}(\beta) \\ &= \hat{T}(-\alpha - \beta)\hat{M}(-\eta, -\eta')\hat{T}(\alpha + \beta)\hat{M}(\eta, \eta') \exp\left(\hat{J}(\eta\cdot\xi' - \eta'\cdot\xi)\right), \end{aligned} \quad (4.8)$$

we may rewrite the generalized Weyl system as

$$\hat{T}(-\alpha)\hat{T}(-\beta)\hat{T}(\alpha)\hat{T}(\beta) = \hat{M}(\eta', \eta)^{-1}\hat{M}(\eta, \eta') \exp\left(\hat{J}(\eta\cdot\xi' - \eta'\cdot\xi)\right), \quad (4.9)$$

similar to (2.3).

5 Monopole quantization/dequantization maps

Next we exhibit generalized Weyl and Wigner maps, associating to the classical observables operators on the quaternionic Hilbert module and viceversa; then, following the deformation quantization programme, we introduce a noncommuting \star -product for the algebra of operator symbols (complex-valued functions). To a complex function on phase space $f(x, p) = f_r(x, p) + if_i(x, p)$, where f_r and f_i are real, let us associate a quaternion by just substituting the quaternionic coordinate-dependent unit $j(x)$ for the imaginary unity i ; so that

$$f \longrightarrow \mathbf{f}(x, p) = f_r(x, p)\mathbf{e}_0 + j(x)f_i(x, p). \quad (5.1)$$

This map is obviously invertible with inverse map

$$f(x, p) = \frac{1}{2}[\text{tr } \mathbf{f} - \mathbf{i} \text{tr } (\mathbf{j}(x)\mathbf{f})]; \quad (5.2)$$

Here tr is the quaternionic trace: with quaternions represented as 2×2 Pauli matrices, $\text{tre}_0 = 2$ and $\text{tre}_i = 0$. The quaternionic-valued operator

$$\hat{\mathbf{f}} = \frac{1}{(2\pi)^3} \int dx \, dp \, d\eta \, d\xi \, e^{-\hat{\mathbf{j}}(\xi x + \eta p)} e^{\hat{\mathbf{j}}(\xi \hat{\mathbf{X}} + \eta \hat{\mathbf{P}})} (f_r(x, p) + \hat{\mathbf{j}} f_i(x, p)) \equiv \hat{\mathbf{f}}_r + \hat{\mathbf{j}} \hat{\mathbf{f}}_i \quad (5.3)$$

is of the form

$$\hat{\mathbf{f}}_{r,i} = \int dx \, dp \, f_{r,i}(x, p) \hat{\Omega}(x, p), \quad (5.4)$$

where we read the *quantizer* $\hat{\Omega}$ off (5.3):

$$\hat{\Omega}(x, p) = \int d\eta \, d\xi \, e^{-\hat{\mathbf{j}}(\xi \cdot x + \eta \cdot p)} \hat{\mathbf{T}}(\alpha) = \int d\eta \, d\xi \, e^{-\hat{\mathbf{j}}(\xi \cdot x + \eta \cdot p)} e^{\hat{\mathbf{j}}[\eta \cdot \hat{\mathbf{P}} + \xi \cdot \hat{\mathbf{X}}]}. \quad (5.5)$$

This is to be compared with the canonical one (2.5).

Now we claim that the dequantization map is given by:

$$\begin{aligned} f(x, p) &= \frac{1}{2} \text{tr} \int d\xi \, d\eta \, \text{Tr}_{op} e^{\hat{\mathbf{j}}(\xi x + \eta p)} e^{-\hat{\mathbf{j}}(\xi \hat{\mathbf{X}} + \eta \hat{\mathbf{P}})} \hat{\mathbf{f}} \\ &\quad - \frac{\mathbf{i}}{2} \text{tr} \int d\xi \, d\eta \, \text{Tr}_{op} e^{\hat{\mathbf{j}}(\xi \cdot x + \eta \cdot p)} e^{-\hat{\mathbf{j}}(\xi \hat{\mathbf{X}} + \eta \hat{\mathbf{P}})} \hat{\mathbf{j}} \hat{\mathbf{f}}, \end{aligned} \quad (5.6)$$

conveniently rewritten as

$$f(x, p) = \frac{1}{2} \text{tr} \text{Tr}_{op} \hat{\mathbf{f}} \hat{\Gamma}(x, p) - \frac{\mathbf{i}}{2} \text{tr} \text{Tr}_{op} \hat{\mathbf{j}} \hat{\mathbf{f}} \hat{\Gamma}(x, p), \quad (5.7)$$

with the quantization being self-dual in that the dequantizer actually is equal to the quantizer:

$$\hat{\Gamma}(x, p) = \hat{\Omega}(x, p). \quad (5.8)$$

We need to prove that (5.6) is indeed the inverse of (5.3). This amounts to show that

$$\text{tr} \text{Tr}_{op} \hat{\Omega}(x, p) \hat{\Gamma}(x', p') = \delta(x - x') \delta(p - p') \quad (5.9)$$

$$\text{tr} \text{Tr}_{op} \hat{\mathbf{j}} \hat{\Omega}(x, p) \hat{\Gamma}(x', p') = 0 \quad (5.10)$$

For this we refer to Appendix A.

5.1 The \star -product

We can now proceed to define the star product as in (2.7)

$$f \star g(x, p) = \frac{1}{2} \left(\text{tr} \text{Tr}_{op} \hat{\mathbf{f}} \hat{\mathbf{g}} \hat{\Gamma}(x, p) - \mathbf{i} \text{tr} \text{Tr}_{op} \hat{\mathbf{j}} \hat{\mathbf{f}} \hat{\mathbf{g}} \hat{\Gamma}(x, p) \right), \quad (5.11)$$

As for the integral kernel, we define, analogously to (2.7)

$$\begin{aligned} K_1(x', p'; x'', p''; x, p) &= \frac{1}{2} \text{tr} \text{Tr}_{op} \hat{\Omega}(x', p') \hat{\Omega}(x'', p'') \hat{\Gamma}(x, p). \\ K_2(x', p'; x'', p''; x, p) &= \frac{1}{2} \text{tr} \text{Tr}_{op} \hat{J} \hat{\Omega}(x', p') \hat{\Omega}(x'', p'') \hat{\Gamma}(x, p). \\ K_M(x', p'; x'', p''; x, p) &= K_1(x', p'; x'', p''; x, p) - i K_2(x', p'; x'', p''; x, p). \end{aligned}$$

With this notation, the star product acquires the usual integral kernel form

$$f \star g(x, p) = \int dx' dp' dx'' dp'' f(x', p') g(x'', p'') K_M(x', p'; x'', p''; x, p) \quad (5.12)$$

The explicit expression of K_M can be calculated observing that

$$\begin{aligned} K_1(x', p'; x'', p''; x, p) &= \text{tr} [\exp(2j(x + x' - x'')((x' - x)p'' + (x'' - x')p + (x - x'')p)) \\ &\quad \mathbf{m}(2(x' - x), 2(x'' - x'); x - x'' + x')] \\ K_2(x', p'; x'', p''; x, p) &= \text{tr} [\exp(2j(x + x' - x'')((x' - x)p'' + (x'' - x')p + (x - x'')p)) \\ &\quad \mathbf{m}(2(x' - x), 2(x'' - x'); x - x'' + x') j(x + x' - x'')] \end{aligned} \quad (5.13)$$

with $\mathbf{m}(2(x' - x), 2(x'' - x'); x - x'' + x')$ from (3.19), given by

$$\mathbf{m} = \exp \left(\frac{g}{|x + x' - x''|^3} j(x + x' - x'') \epsilon_{ijk} (x + x' - x'')^i (x' - x)^j (x'' - x')^k \right). \quad (5.14)$$

We have then for $K_M(x', p'; x'', p''; x, p)$ the expression

$$K_M = C \exp \left[2i \left((x' - x) \cdot p'' + (x'' - x') \cdot p + (x - x'') \cdot p' + \frac{g}{2} \frac{x \cdot (x' \wedge x'')}{|x - x'' + x'|^3} \right) \right] \quad (5.15)$$

and C is a normalization constant. In the case $g = 0$ we recover the result for the Moyal kernel (2.7).

We may use this result to compute the star product of the coordinate functions in phase space. Products which involve at least one coordinate function x^i are easy to calculate, less trivial is the computation of the star product of momenta. We find:

$$x^i \star x^j = x^i x^j \quad (5.16)$$

$$x^i \star p_j = x^i p_j - \frac{i}{2} \delta_j^i \quad (5.17)$$

$$p_i \star p_j = p_i p_j - \frac{i}{4} g \epsilon_{ijk} \frac{x^k}{|x|^3} \quad (5.18)$$

This obviously reproduces the Poisson structure.

6 Concluding remarks

The conjecture of the authors in [3], that a Weyl–Wigner–Groenewold–Moyal quantization of a phase space in the (asymptotic) type of a 't Hooft–Polyakov monopole can be effected, by choosing as fundamental complex structure the quaternionic operator j studied by Emch and Jadczyk, has been proved.

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A Proof of (5.9), (5.10)

As for (5.9)

$$\begin{aligned}
& \text{tr Tr}_{op} \hat{\Omega}(x, p) \hat{\Gamma}(x', p') = \int d\eta \, d\xi \, d\eta' \, d\xi' \text{tr Tr}_{op} \hat{\Omega}(x, p) \hat{\Gamma}(x', p') \\
&= \int d\eta \, d\xi \, d\eta' \, d\xi' \text{tr Tr}_{op} \left[\hat{T}(\alpha) \hat{T}(\beta) \exp[-\hat{J}(\xi x + \eta p)] \exp[-\hat{J}(\xi' x' + \eta' p')] \right] \\
&= \int d\eta \, d\xi \, d\eta' \, d\xi' \text{tr Tr}_{op} \left[\hat{T}(\alpha + \beta) M(\eta, \eta') e^{\frac{1}{2} \hat{J}(\eta \xi' - \eta' \xi)} \exp[-\hat{J}(\xi x + \eta p)] \exp[-\hat{J}(\xi' x' + \eta' p')] \right] \\
&= \int d\eta \, d\xi \, d\eta' \, d\xi' \text{tr Tr}_{op} \left[e^{\hat{J}(\eta + \eta') \hat{P}} e^{J(\xi + \xi') \hat{X}} e^{-(\eta + \eta')(\xi + \xi')/2} \hat{M}(\eta, \eta') \right. \\
&\quad \left. e^{\frac{1}{2} \hat{J}(\eta \xi' - \eta' \xi)} \exp[-\hat{J}(\xi x + \eta p)] \exp[-\hat{J}(\xi' x' + \eta' p')] \right] \\
&= \int d\eta \, d\xi \, d\eta' \, d\xi' \, dy \text{tr} e^{-(\xi x + \eta p + \xi' x' + \eta' p') j(y)} \delta(\eta + \eta') \mathbf{w}(\eta + \eta', y - \eta - \eta') \\
&\quad \cdot e^{j(y)(\xi + \xi') y} e^{j(y)(\xi + \xi')(\eta + \eta')} \mathbf{m}(\eta, \eta'; y) e^{j(y)(\eta \xi' - \eta' \xi)/2} \\
&= \int d\xi \, d\eta \, d\xi' \, dy \text{tr} e^{-(\xi x + \xi' x' + \eta(p - p')) j(y)} e^{j(y)(\xi + \xi') y} e^{j(y) \eta(\xi + \xi')/2} \\
&= \int d\xi \, d\xi' \, dy \, \text{tr} e^{-(\xi x + \xi' x') j(y)} e^{j(y)(\xi + \xi') y} \delta\left(\frac{\xi' + \xi}{2} - (p - p')\right) \\
&= \int d\xi \, dy \text{tr} e^{-[(\xi - x')x + 2(p - p')x'] j(y)} e^{j(y) 2(p - p') y} \\
&= \int dy \, \text{tr} e^{-2(p - p') x' j(y)} e^{j(y) 2(p - p') y} \delta(x - x') = \int dy \, \text{tr} e^{j(y) 2(p - p')(x' - y)} \delta(x - x') \\
&= 2 \int dy \, \cos[2(p - p')(x' - y)] \delta(x - x') = \delta(p - p') \delta(x - x'). \tag{A.1}
\end{aligned}$$

The proof of (5.10) is similar.

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